A parameter–uniform finite difference method for a singularly perturbed linear system of second order ordinary differential equations of reaction-diffusion type

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dedicated to G. I. Shishkin on his 70th birthday

Summary. A singularly perturbed linear system of second order ordinary differential equations of reaction-diffusion type with given boundary conditions is considered. The leading term of each equation is multiplied by a small positive parameter. These parameters are assumed to be distinct. The components of the solution exhibit overlapping layers. Shishkin piecewise-uniform meshes are introduced, which are used in conjunction with a classical finite difference discretisation, to construct two numerical methods for solving this problem. It is proved that the numerical approximations obtained with these methods are essentially first, respectively second, order convergent uniformly with respect to all of the parameters.

1 Introduction

The following two-point boundary value problem is considered for the singularly perturbed linear system of second order differential equations

$$-E\mathbf{u}''(x)+A(x)\mathbf{u}(x)=\mathbf{f}(x), \quad x\in(0,1), \quad \mathbf{u}(0) \text{ and } \mathbf{u}(1) \text{ given.} \tag{1}$$

Here $\mathbf{u}$ is a column $n-$vector, E and $A(x)$ are $n\times n$ matrices, $E=\mathrm{diag}(\boldsymbol{\varepsilon})$, $\boldsymbol{\varepsilon}=(\varepsilon_1,\,\cdots,\,\varepsilon_n)$ with $0<\varepsilon_i\leq 1$ for all $i=1,\ldots,n$. The ε_i are assumed to be distinct and, for convenience, to have the ordering

$$\varepsilon_1 < \cdots < \varepsilon_n.$$

Cases with some of the parameters coincident are not considered here. The problem can also be written in the operator form

$$\mathbf{Lu} \ = \ \mathbf{f}, \quad \mathbf{u}(0) \text{ and } \mathbf{u}(1) \text{ given}$$

where the operator $\mathbf{L}$ is defined by

$$\mathbf{L} \ = \ -ED^2 + A(x) \quad \text{and} \quad D^2 = \frac{d^2}{dx^2}.$$

For all $x \in [0,1]$ it is assumed that the components $a_{ij}(x)$ of $A(x)$ satisfy the inequalities

$$a_{ii}(x) > \sum_{\substack{j\neq i\\ j=1}}^{n} |a_{ij}(x)| \ \text{ for } \ 1 \leq \mathrm{i} \leq \mathrm{n}, \ \text{ and } \ \mathrm{a_{ij}(x)} \leq 0 \ \text{ for } \ \mathrm{i} \neq \mathrm{j} \tag{2}$$

and, for some α,

$$0 < \alpha < \min_{\substack{x\in[0,1]\\ 1\leq i\leq n}} \left(\sum_{j=1}^{n} a_{ij}(x)\right). \tag{3}$$

Wherever necessary the required smoothness of the problem data is assumed. It is also assumed, without loss of generality, that

$$\max_{1\leq i\leq n} \sqrt{\varepsilon_i} \leq \frac{\sqrt{\alpha}}{4}. \tag{4}$$

The norms $\parallel \mathbf{V} \parallel = \max_{1\leq k\leq n} |V_k|$ for any n-vector $\mathbf{V}$, $\parallel y \parallel = \sup_{0\leq x\leq 1} |y(x)|$ for any scalar-valued function y and $\parallel \mathbf{y} \parallel = \max_{1\leq k\leq n} \parallel y_k \parallel$ for any vector-valued function $\mathbf{y}$ are introduced. Throughout the paper C denotes a generic positive constant, which is independent of x and of all singular perturbation and discretization parameters. Furthermore, inequalities between vectors are understood in the componentwise sense.

For a general introduction to parameter-uniform numerical methods for singular perturbation problems, see [1], [2] and [4]. Parameter-uniform numerical methods for various special cases of (1) are examined in, for example, [5], [6] and [7]. For (1) itself parameter-uniform numerical methods of first and second order are considered in [8]. However, the present paper differs from [8] in two important ways. First of all, the meshes, and hence the numerical methods, used are different from those in [8]; the transition points between meshes of differing resolution are defined in a similar but different manner. The piecewise-uniform Shishkin meshes $M_{\mathbf{b}}$ in the present paper have the elegant property that they reduce to uniform meshes whenever $\mathbf{b} = \mathbf{0}$. Secondly, the proofs of essentially first and second order parameter-uniform convergence

do not require the use of Green's function techniques, as is the case in [8]. The significance of this is that it is more likely that such techniques can be extended in future to problems in higher dimensions and to nonlinear problems, than is the case for proofs depending on Green's functions. It is also satisfying, and appropriate in this special issue, to be able to demonstrate that the methods of proof pioneered by G. I. Shishkin can be extended successfully to problems of this kind.

The plan of the paper is as follows. In the next section both standard and novel bounds on the smooth and singular components of the exact solution are obtained. The sharp estimates for the singular component in Lemma 5 are proved by mathematical induction, while an interesting ordering of the points $x_{i,j}$ is established in Lemma 6. In Section 3 appropriate piecewise-uniform Shishkin meshes for essentially first order numerical methods are introduced, the discrete problem is defined and the discrete maximum principle and discrete stability properties are established. In Section 4 an expression for the local truncation error is found and two distinct standard estimates are stated. In Section 5 parameter-uniform estimates for the local truncation error of the smooth and singular components are obtained in a sequence of lemmas. The section culminates with the statement and proof of the essentially first order parameter-uniform error estimate. In the final section an outline of the construction and error estimation of an essentially second order parameter-uniform numerical method is presented.

2 Analytical results

The operator $\mathbf{L}$ satisfies the following maximum principle

Lemma 1. *Let $A(x)$ satisfy (2) and (3). Let $\boldsymbol{\psi}$ be any function in the domain of $\mathbf{L}$ such that $\boldsymbol{\psi}(0) \geq \mathbf{0}$ and $\boldsymbol{\psi}(1) \geq \mathbf{0}$. Then $\mathbf{L}\boldsymbol{\psi}(x) \geq \mathbf{0}$ for all $x \in (0,1)$ implies that $\boldsymbol{\psi}(x) \geq \mathbf{0}$ for all $x \in [0,1]$.*

Proof. Let i^*, x^* be such that $\psi_{i^*}(x^*) = \min_{i,x} \psi_i(x)$ and assume that the lemma is false. Then $\psi_{i^*}(x^*) < 0$. From the hypotheses we have $x^* \not\in \{0,1\}$ and $\psi''_{i^*}(x^*) \geq 0$. Thus

$$(\mathbf{L}\boldsymbol{\psi}(x^*))_{i^*} = -\varepsilon_{i^*}\psi''_{i^*}(x^*) + \sum_{j=1}^{n} a_{i^*,j}(x^*)\psi_j(x^*) < 0,$$

which contradicts the assumption and proves the result for $\mathbf{L}$. ∎

Let $\tilde{A}(x)$ be any principal sub-matrix of $A(x)$ and $\tilde{\mathbf{L}}$ the corresponding operator. To see that any $\tilde{\mathbf{L}}$ satisfies the same maximum principle as $\mathbf{L}$, it suffices to observe that the elements of $\tilde{A}(x)$ satisfy *a fortiori* the same inequalities as those of $A(x)$.

We remark that the maximum principle is not necessary for the results that follow, but it is a convenient tool in their proof.

Lemma 2. *Let $A(x)$ satisfy (2) and (3). If $\boldsymbol{\psi}$ is any function in the domain of $\mathbf{L}$, then for each i, $1 \le i \le n$,*

$$|\boldsymbol{\psi}_i(x)| \le \max\left\{\parallel \boldsymbol{\psi}(0) \parallel, \parallel \boldsymbol{\psi}(1) \parallel, \frac{1}{\alpha} \parallel \mathbf{L}\boldsymbol{\psi} \parallel\right\}, \qquad x \in [0,1].$$

Proof. Define the two functions

$$\boldsymbol{\theta}^{\pm}(x) \ = \ \max\left\{\parallel \boldsymbol{\psi}(0) \parallel, \ \parallel \boldsymbol{\psi}(1) \parallel, \ \frac{1}{\alpha} \parallel \mathbf{L}\boldsymbol{\psi} \parallel\right\}\mathbf{e} \ \pm \ \boldsymbol{\psi}(x)$$

where $\mathbf{e} \ = \ (1, \ \ldots, \ 1)^T$ is the unit column vector. Using the properties of A it is not hard to verify that $\boldsymbol{\theta}^{\pm}(0) \ \ge \ \mathbf{0}$, $\boldsymbol{\theta}^{\pm}(1) \ \ge \ \mathbf{0}$ and $\mathbf{L}\boldsymbol{\theta}^{\pm}(x) \ \ge \ \mathbf{0}$. It follows from Lemma 1 that $\boldsymbol{\theta}^{\pm}(x) \ \ge \ \mathbf{0}$ for all $x \ \in \ [0, \ 1]$. ∎

A standard estimate of the exact solution and its derivatives is contained in the following lemma.

Lemma 3. *Let $A(x)$ satisfy (2) and (3) and let $\mathbf{u}$ be the exact solution of (1). Then, for each $i = 1 \ \ldots \ n$, all $x \in [0,1]$ and $k = 0, 1, 2$,*

$$|u_i^{(k)}(x)| \le C\varepsilon_i^{-\frac{k}{2}}(||\mathbf{u}(0)|| + ||\mathbf{u}(1)|| + ||\mathbf{f}||)$$

and

$$|u_i'''(x)| \le C\varepsilon_i^{-\frac{3}{2}}(||\mathbf{u}(0)|| + ||\mathbf{u}(1)|| + ||\mathbf{f}|| + ||\mathbf{f}'||)$$

Proof. The bound on $\mathbf{u}$ is an immediate consequence of Lemma 2 and the differential equation.
To bound $u_i'(x)$, for all i and any x, consider an interval $N_x = [a, a + \sqrt{\varepsilon_i}]$ such that $x \in N_x$. Then, by the mean value theorem, for some $y \in N_x$,

$$u_i'(y) = \frac{u_i(a + \sqrt{\varepsilon_i}) - u_i(a)}{\sqrt{\varepsilon_i}}$$

and it follows that

$$|u_i'(y)| \le 2\varepsilon_i^{-\frac{1}{2}}||u_i||.$$

Now

$$\mathbf{u}'(x) = \mathbf{u}'(y) + \int_y^x \mathbf{u}''(s)ds = \mathbf{u}'(y) + E^{-1}\int_y^x (-\mathbf{f}(s) + A(s)\mathbf{u}(s))ds$$

and so

$$|u_i'(x)| \le |u_i'(y)| + C\varepsilon_i^{-1}(||f_i|| + ||\mathbf{u}||)\int_y^x ds \le C\varepsilon_i^{-\frac{1}{2}}(||f_i|| + ||\mathbf{u}||)$$

from which the required bound follows.
Rewriting and differentiating the differential equation gives $\mathbf{u}'' = E^{-1}(A\mathbf{u} - \mathbf{f})$, $\mathbf{u}''' = E^{-1}(A\mathbf{u}' + A'\mathbf{u} - \mathbf{f}')$, and the bounds on u_i'', u_i''' follow. ∎

The reduced solution $\mathbf{u}_0$ of (1) is the solution of the reduced equation $A\mathbf{u}_0 = \mathbf{f}$. The Shishkin decomposition of the exact solution $\mathbf{u}$ of (1) is $\mathbf{u} = \mathbf{v} + \mathbf{w}$ where the smooth component $\mathbf{v}$ is the solution of $\mathbf{Lv} = \mathbf{f}$ with $\mathbf{v}(0) = \mathbf{u}_0(0)$ and $\mathbf{v}(1) = \mathbf{u}_0(1)$ and the singular component $\mathbf{w}$ is the solution of $\mathbf{Lw} = \mathbf{0}$ with $\mathbf{w}(0) = \mathbf{u}(0) - \mathbf{v}(0)$ and $\mathbf{w}(1) = \mathbf{u}(1) - \mathbf{v}(1)$. For convenience the left and right boundary layers of $\mathbf{w}$ are separated using the further decomposition $\mathbf{w} = \mathbf{w}^l + \mathbf{w}^r$ where $\mathbf{Lw}^l = \mathbf{0},\ \mathbf{w}^l(0) = \mathbf{u}(0) - \mathbf{v}(0),\ \mathbf{w}^l(1) = \mathbf{0}$ and $\mathbf{Lw}^r = \mathbf{0},\ \mathbf{w}^r(0) = \mathbf{0},\ \mathbf{w}^r(1) = \mathbf{u}(1) - \mathbf{v}(1)$.
Bounds on the smooth component and its derivatives are contained in

Lemma 4. *Let $A(x)$ satisfy (2) and (3). Then the smooth component $\mathbf{v}$ and its derivatives satisfy, for all $x \in [0,1]$, $i = 1, \dots n$ and $k = 0, \dots 3$,*

$$|v_i^{(k)}(x)| \leq C(1 + \varepsilon_i^{1-\frac{k}{2}}).$$

Proof. The bound on $\mathbf{v}$ is an immediate consequence of the defining equations for $\mathbf{v}$ and Lemma 2.
The bounds on $\mathbf{v}'$ and $\mathbf{v}''$ are found as follows. Differentiating twice the equation for $\mathbf{v}$, it is not hard to see that $\mathbf{v}''$ satisfies

$$\mathbf{Lv}'' = \mathbf{g}, \quad \text{where} \ \ \mathbf{g} = \mathbf{f}'' - A''\mathbf{v} - 2A'\mathbf{v}'.$$

Also the defining equations for $\mathbf{v}$ yield at $x = 0,\ \ x = 1$

$$\mathbf{v}''(0) = \mathbf{0}, \ \ \mathbf{v}''(1) = \mathbf{0}.$$

Applying Lemma 2 to $\mathbf{v}''$ then gives

$$||\mathbf{v}''|| \leq C(1 + ||\mathbf{v}'||). \tag{5}$$

Choosing $i^*,\ x^*$, such that $1 \leq i^* \leq n,\ x^* \in (0,1)$ and

$$v'_{i^*}(x^*) = ||\mathbf{v}'|| \tag{6}$$

and using a Taylor expansion it follows that, for any $y \in [0, 1 - x^*]$ and some $\eta,\ x^* < \eta < x^* + y$,

$$v_{i^*}(x^* + y) = v_{i^*}(x^*) + y\ \mathbf{v}'(x^*) + \frac{y^2}{2}\ v''_{i^*}(\eta). \tag{7}$$

Rearranging (7) yields

$$v'_{i^*}(x^*) = \frac{v_{i^*}(x^* + y) - v_{i^*}(x^*)}{y} - \frac{y}{2} v''_{i^*}(\eta) \tag{8}$$

and so, from (6) and (8),

$$||\mathbf{v}'|| \leq \frac{2}{y}||\mathbf{v}|| + \frac{y}{2}||\mathbf{v}''||. \tag{9}$$

Using (9), (5) and the bound on $\mathbf{v}$ yields

$$(1-\frac{Cy}{2})||\mathbf{v}''|| \le C(1+\frac{2}{y}). \tag{10}$$

Choosing $y=\min(\frac{1}{C},1-x^*)$, (10) then gives $||\mathbf{v}''|| \le C$ and (9) gives $||\mathbf{v}'|| \le C$ as required. The bound on $\mathbf{v}'''$ is obtained by a similar argument. ∎

The layer functions $B_i^l,\ B_i^r,\ i=1,\ \dots,\ n,$, associated with the solution $\mathbf{u}$, are defined on $[0,1]$ by

$$B_i^l(x)=e^{-x\sqrt{\alpha/\varepsilon_i}},\ B_i^r(x)=B_i^l(1-x).$$

The following elementary properties of these layer functions, for all $1\le i<j\le n$ and $0\le x<y\le 1$, should be noted:
(i) $B_i^l(x)\ <\ B_j^l(x),\ \ B_i^l(x)\ >\ B_i^l(y),\ \ 0\ <\ B_i^l(x)\ \le\ 1.$
(ii) $B_i^r(x)\ <\ B_j^r(x),\ \ B_i^r(x)\ <\ B_i^r(y),\ \ 0\ <\ B_i^r(x)\ \le\ 1.$

Bounds on the singular components $\mathbf{w}^l,\ \mathbf{w}^r$ of $\mathbf{u}$ and their derivatives are contained in

Lemma 5. *Let $A(x)$ satisfy (2) and (3).Then there exists a constant C, such that, for each $x\in[0,1]$ and $i=1,\ \dots,\ n$,*

$$\left|w_i^l(x)\right|\ \le\ CB_n^l(x),\ \ \left|w_i^{l,\prime}(x)\right|\ \le\ C\sum_{q=i}^{n}\frac{B_q^l(x)}{\sqrt{\varepsilon_q}},$$

$$\left|w_i^{l,\prime\prime}(x)\right|\ \le\ C\sum_{q=i}^{n}\frac{B_q^l(x)}{\varepsilon_q},\ \ \left|\varepsilon_i w_i^{l,\prime\prime\prime}(x)\right|\ \le\ C\sum_{q=1}^{n}\frac{B_q^l(x)}{\sqrt{\varepsilon_q}}.$$

Analogous results hold for w_i^r and its derivatives.

Proof. First we obtain the bound on $\mathbf{w}^l$. We define the two functions $\boldsymbol{\theta}^{\pm}=CB_n^l\mathbf{e}\pm\mathbf{w}^l$. Then clearly $\boldsymbol{\theta}^{\pm}(0)\ge\mathbf{0}$, $\boldsymbol{\theta}^{\pm}(1)\ge\mathbf{0}$ and $L\boldsymbol{\theta}^{\pm}=CL(B_n^l\mathbf{e})$. Then, for $i=1,\dots,n$, $(L\boldsymbol{\theta}^{\pm})_i=C(\sum_{j=1}^{n}a_{i,j}-\alpha\frac{\varepsilon_i}{\varepsilon_n})B_n^l>0$. By Lemma 1, $\boldsymbol{\theta}^{\pm}\ge\mathbf{0}$, which leads to the required bound on $\mathbf{w}^l$.

Assuming, for the moment, the bounds on the first derivatives $w_i^{l,\prime}$, the system of differential equations satisfied by $\mathbf{w}^l$ is differentiated to get

$$-E\mathbf{w}^{l,\prime\prime\prime}+A\mathbf{w}^{l,\prime}+A'\mathbf{w}^l=\mathbf{0}.$$

The required bounds on the $w_i^{l,\prime\prime\prime}$ follow from those on w_i^l and $w_i^{l,\prime}$. It remains therefore to establish the bounds on $w_i^{l,\prime}$ and $w_i^{l,\prime\prime}$, for which the following mathematical induction argument is used. It is assumed that the bounds hold for all systems up to order $n-1$. It is then shown that the bounds hold for order n. The induction argument is completed by observing that the bounds for the scalar case $n=1$ are proved in [1].

It is now shown that under the induction hypothesis the required bounds hold for $w_i^{l,\prime}$ and $w_i^{l,\prime\prime}$. The bounds when i=n are established first.The differential equation for w_n^l gives $\varepsilon_n w_n^{l,\prime\prime} = (A\mathbf{w}^l)_n$ and the required bound on $w_n^{l,\prime\prime}$ follows at once from that for $\mathbf{w}^l$. For $w_n^{l,\prime}$ it is seen from the bounds in Lemma 3, applied to the system satisfied by $\mathbf{w}^l$, that $|w_i^{l,\prime}(x)| \le C\varepsilon_i^{-\frac{1}{2}}$. In particular, $|w_n^{l,\prime}(0)| \le C\varepsilon_n^{-\frac{1}{2}}$ and $|w_n^{l,\prime}(1)| \le C\varepsilon_n^{-\frac{1}{2}}$. It is also not hard to verify that $\mathbf{L}\mathbf{w}^{l,\prime} = -A'\mathbf{w}^l$. Using these results, the inequalities $\varepsilon_i < \varepsilon_n,\ i < n$, and the properties of A, it follows that the two barrier functions $\boldsymbol{\theta}^{\pm} = CE^{-\frac{1}{2}}B_n^l\mathbf{e}\pm\mathbf{w}^{l,\prime}$ satisfy the inequalities $\boldsymbol{\theta}^{\pm}(0) \ge \mathbf{0},\ \boldsymbol{\theta}^{\pm}(1) \ge \mathbf{0}$, and $\mathbf{L}\boldsymbol{\theta}^{\pm} \ge \mathbf{0}$. It follows from Lemma 1 that $\boldsymbol{\theta}^{\pm} \ge \mathbf{0}$ and in particular that its n^{th} component satisfies $|w_n^{l,\prime}(x)| \le C\varepsilon_n^{-\frac{1}{2}}B_n^l(x)$ as required.

To bound $w_i^{l,\prime}$ and $w_i^{l,\prime\prime}$ for $1 \le i \le n-1$ introduce $\tilde{\mathbf{w}}^l = (w_1^l, \dots, w_{n-1}^l)$. Then, taking the first $n-1$ equations satisfied by $\mathbf{w}^l$, it follows that

$$-\tilde{E}\tilde{\mathbf{w}}^{l,\prime\prime} + \tilde{A}\tilde{\mathbf{w}}^l = \mathbf{g},$$

where $\tilde{E},\ \tilde{A}$ is the matrix obtained by deleting the last row and column from $E,\ A$, respectively, and the components of $\mathbf{g}$ are $g_i = -a_{i,n}w_n^l$ for $1 \le i \le n-1$. Using the bounds already obtained for $w_n^l, w_n^{l,\prime}$, it is seen that $\mathbf{g}$ is bounded by $CB_n^l(x)$ and $\mathbf{g}'$ by $C\frac{B_n^l(x)}{\sqrt{\varepsilon_n}}$. The boundary conditions for $\tilde{\mathbf{w}}^l$ are $\tilde{\mathbf{w}}^l(0) = \tilde{\mathbf{u}}(0) - \tilde{\mathbf{u}}^0(0)$, $\tilde{\mathbf{w}}^l(1) = \mathbf{0}$, where $\mathbf{u}^0$ is the solution of the reduced problem $\mathbf{u}^0 = A^{-1}\mathbf{f}$, and are bounded by $C(\| \mathbf{u}(0) \| + \| \mathbf{f}(0) \|)$ and $C(\| \mathbf{u}(1) \| + \| \mathbf{f}(1) \|)$. Now decompose $\tilde{\mathbf{w}}^l$ into smooth and singular components to get

$$\tilde{\mathbf{w}}^l = \mathbf{q} + \mathbf{r}, \quad \tilde{\mathbf{w}}^{l,\prime} = \mathbf{q}' + \mathbf{r}'.$$

Applying Lemma 4 to $\mathbf{q}$ and using the bounds on the inhomogeneous term $\mathbf{g}$ and its derivative $\mathbf{g}'$, it follows that $|\mathbf{q}'(x)| \le C\frac{B_n^l(x)}{\sqrt{\varepsilon_n}}$ and $|\mathbf{q}''(x)| \le C\frac{B_n^l(x)}{\varepsilon_n}$. Using mathematical induction, assume that the result holds for all systems with $n-1$ equations. Then Lemma 5 applies to $\mathbf{r}$ and so, for $i = 1, \dots, n-1$,

$$|r_i'(x)| \le C\sum_{q=i}^{n-1}\frac{B_q^l(x)}{\sqrt{\varepsilon_q}},\ |r_i''(x)| \le C\sum_{q=i}^{n-1}\frac{B_q^l(x)}{\varepsilon_q}.$$

Combining the bounds for the derivatives of q_i and r_i, it follows that

$$|w_i^{l,\prime}(x)| \le C\sum_{q=i}^{n-1}\frac{B_q^l(x)}{\sqrt{\varepsilon_q}},\ |w_i^{l,\prime\prime}(x)| \le C\sum_{q=i}^{n-1}\frac{B_q^l(x)}{\varepsilon_q}.$$

Thus, the bounds on $w_i^{l,\prime}$ and $w_i^{l,\prime\prime}$ hold for a system with n equations, as required. The proof of the analogous results for the right boundary layer functions is analogous. ∎

Definition 1. *For B_i^l, B_j^l and each $1 \le i \ne j \le n$, the point $x_{i,j}$ is defined by*

$$\frac{B^l_i(x_{i,j})}{\sqrt{\varepsilon_i}} = \frac{B^l_j(x_{i,j})}{\sqrt{\varepsilon_j}}. \tag{11}$$

It is remarked that

$$\frac{B^r_i(1-x_{i,j})}{\sqrt{\varepsilon_i}} = \frac{B^r_j(1-x_{i,j})}{\sqrt{\varepsilon_j}}. \tag{12}$$

In the next lemma it is shown that the points $x_{i,j}$ exist, are uniquely defined, lie in the domain $[0, \frac{1}{2}]$ and have an interesting ordering.

Lemma 6. *Assume that (4) holds. If, in addition, $\sqrt{\varepsilon_i} \le \sqrt{\varepsilon_j}/2$, then, for all i, j with $1 \le i < j \le n$, the points $x_{i,j}$ exist, are uniquely defined, lie in $(0, \frac{1}{2}]$ and satisfy the following inequalities*

$$\frac{B^l_i(x)}{\sqrt{\varepsilon_i}} > \frac{B^l_j(x)}{\sqrt{\varepsilon_j}},\ \ x \in [0, x_{i,j}),\quad \frac{B^l_i(x)}{\sqrt{\varepsilon_i}} < \frac{B^l_j(x)}{\sqrt{\varepsilon_j}},\ x \in (x_{i,j}, 1]. \tag{13}$$

In addition the following ordering holds

$$x_{i,j} < x_{i+1,j},\ \text{if}\ \ i+1 < j\ \ \text{and}\ \ x_{i,j} < x_{i,j+1},\ \ \text{if}\ \ i < j. \tag{14}$$

Analogous results hold for the B^r_i, B^r_j and the points $1 - x_{i,j}$.

Proof. Existence, uniqueness and (13) follow from the observation that $\sqrt{\varepsilon_i} < \sqrt{\varepsilon_j}$, for $i < j$, and the ratio of the two sides of (11), namely

$$\frac{B^l_i(x)}{\sqrt{\varepsilon_i}} \frac{\sqrt{\varepsilon_j}}{B^l_j(x)} = \frac{\sqrt{\varepsilon_j}}{\sqrt{\varepsilon_i}} \exp\left(-\sqrt{\alpha}x\left(\frac{1}{\sqrt{\varepsilon_i}} - \frac{1}{\sqrt{\varepsilon_j}}\right)\right),$$

is monotonically decreasing from the value $\frac{\sqrt{\varepsilon_j}}{\sqrt{\varepsilon_i}} > 1$ as x increases from 0. The point $x_{i,j}$ is the unique point x at which this ratio has the value 1. Rearranging (11) gives

$$x_{i,j} = \frac{\ln(\frac{1}{\sqrt{\varepsilon_i}}) - \ln(\frac{1}{\sqrt{\varepsilon_j}})}{\sqrt{\alpha}(\frac{1}{\sqrt{\varepsilon_i}} - \frac{1}{\sqrt{\varepsilon_j}})} = \frac{\ln(\frac{\sqrt{\varepsilon_j}}{\sqrt{\varepsilon_i}})}{\sqrt{\alpha}(\frac{1}{\sqrt{\varepsilon_i}} - \frac{1}{\sqrt{\varepsilon_j}})}. \tag{15}$$

Using the hypotheses it follows that

$$x_{i,j} < \frac{2\sqrt{\varepsilon_i}}{\sqrt{\alpha}} \ln\left(\frac{\sqrt{\varepsilon_j}}{\sqrt{\varepsilon_i}}\right) < \frac{2\sqrt{\varepsilon_i}}{\sqrt{\alpha}} \frac{\sqrt{\varepsilon_j}}{\sqrt{\varepsilon_i}} = \frac{2\sqrt{\varepsilon_j}}{\sqrt{\alpha}} < \frac{1}{2}$$

as required.
To prove (14), returning to (15) and writing $\varepsilon_k = \exp(-p_k)$, for some $p_k > 0$ and all k, it follows that

$$x_{i,j} = \frac{p_i - p_j}{\sqrt{\alpha}(\exp p_i - \exp p_j)}.$$

The inequality $x_{i,j} < x_{i+1,j}$ is equivalent to

$$\frac{p_i - p_j}{\exp p_i - \exp p_j} < \frac{p_{i+1} - p_j}{\exp p_{i+1} - \exp p_j},$$

which can be written in the form

$$(p_{i+1} - p_j)\exp(p_i - p_j) + (p_i - p_{i+1}) - (p_i - p_j)\exp(p_{i+1} - p_j) > 0.$$

With $a = p_i - p_j$ and $b = p_{i+1} - p_j$ it is not hard to see that $a > b > 0$ and $a - b = p_i - p_{i+1}$. Moreover, the previous inequality is then equivalent to

$$\frac{\exp a - 1}{a} > \frac{\exp b - 1}{b},$$

which is true because $a > b$ and proves the first part of (14). The second part is proved by a similar argument.
The analogous results for the B_i^r, B_j^r and the points $1 - x_{i,j}$ are proved by a similar argument.

3 The discrete problem

A piecewise uniform mesh with N mesh-intervals and mesh-points $\{x_i\}_{i=0}^N$ is now constructed by dividing the interval $[0,1]$ into $2n+1$ sub-intervals as follows

$$[0,\sigma_1] \cup \cdots \cup (\sigma_{n-1},\sigma_n] \cup (\sigma_n, 1-\sigma_n] \cup (1-\sigma_n, 1-\sigma_{n-1}] \cup \cdots \cup (1-\sigma_1, 1].$$

The n transition parameters, which determine the points separating the uniform meshes, are defined by

$$\sigma_n = \min\left\{\frac{1}{4}, \sqrt{\frac{\varepsilon_n}{\alpha}}\ln N\right\} \tag{16}$$

and for $i = 1, \ \dots \ , n-1$

$$\sigma_i = \min\left\{\frac{\sigma_{i+1}}{2}, \sqrt{\frac{\varepsilon_i}{\alpha}}\ln N\right\}. \tag{17}$$

Clearly

$$0 \ < \ \sigma_1 \ < \ \dots \ < \ \sigma_n \ \leq \ \frac{1}{4}, \qquad \frac{3}{4} \leq 1 - \sigma_n < \ \dots \ 1 - \sigma_1 < 1.$$

Then, on the sub-interval $(\sigma_n, 1-\sigma_n]$ a uniform mesh with $\frac{N}{2}$ mesh-intervals is placed, on each of the sub-intervals $(\sigma_i, \sigma_{i+1}]$ and $(1-\sigma_{i+1}, 1-\sigma_i]$, $i = 1, \dots, n-1$, a uniform mesh of $\frac{N}{2^{n-i+2}}$ mesh-intervals is placed and on

both of the sub-intervals $[0, \sigma_1]$ and $(1-\sigma_1, 1]$ a uniform mesh of $\frac{N}{2^{n+1}}$ mesh-intervals is placed. In practice it is convenient to take $N = 2^n k$ where k is some positive power of 2. This construction leads to a class of 2^n piecewise uniform Shishkin meshes $M_{\mathbf{b}}$, where $\mathbf{b}$ denotes an n–vector with $b_i = 0$ if $\sigma_i = \frac{\sigma_{i+1}}{2}$ and $b_i = 1$ otherwise. Note that $M_{\mathbf{b}}$ is a classical uniform mesh when $\mathbf{b} = \mathbf{0}$. It is not hard to see also that

$$\sigma_i = 2^{-(j-i+1)}\sigma_{j+1} \text{ when } b_i = \cdots = b_j = 0. \tag{18}$$

$$B_i^l(\sigma_i) = B_i^r(1-\sigma_i) = N^{-1} \text{ when } b_i = 1. \tag{19}$$

Writing $\delta_j = x_{j+1} - x_{j-1}$ note that, on any $M_{\mathbf{b}}$,

$$\delta_j \le CN^{-1}, \quad 1 \le j \le N-1 \tag{20}$$

and

$$\sigma_i \le C\sqrt{\varepsilon_i}\ln N, \quad 1 \le i \le n. \tag{21}$$

Furthermore,

$$B_i^l(\sigma_i - (x_j - x_{j-1})) \le CB_i^l(\sigma_i) \text{ if } x_j = \sigma_i, \text{ for some } i, j. \tag{22}$$

To verify (22) note that if $x_j = \sigma_i$ then $x_j - x_{j-1} = \frac{\sigma_i - \sigma_{i-1}}{(N/2^{n-i+2})} \le N^{-1}\sigma_i 2^{n-i+2}$ and the result follows from $B_i^l(\sigma_i - (x_j - x_{j-1})) \le B_i^l(\sigma_i - \frac{N^{-1}\sigma_i}{2^{n-i+2}}) \le \exp\left(\frac{N^{-1}\ln N}{2^{n-i+2}}\right)B_i^l(\sigma_i)$ and $\exp\left(\frac{N^{-1}\ln N}{2^{n-i+2}}\right) \le C$.
The discrete two-point boundary value problem is now defined on any mesh $M_{\mathbf{b}}$ by the finite difference method

$$-E\delta^2\mathbf{U} + A(x)\mathbf{U} = \mathbf{f}(x), \qquad \mathbf{U}(0) = \mathbf{u}(0), \ \ \mathbf{U}(1) = \mathbf{u}(1). \tag{23}$$

This is used to compute numerical approximations to the exact solution of (1). Note that (23) can also be written in the operator form

$$\mathbf{L}^N\mathbf{U} \ = \ \mathbf{f}, \qquad \mathbf{U}(0) = \mathbf{u}(0), \ \ \mathbf{U}(1) = \mathbf{u}(1)$$

where

$$\mathbf{L}^N \ = \ -E\delta^2 + A(x)$$

and δ^2, D^+ and D^- are the difference operators

$$\delta^2\mathbf{U}(x_j) \ = \ \frac{D^+\mathbf{U}(x_j) - D^-\mathbf{U}(x_j)}{\overline{h}_j}$$

$$D^+\mathbf{U}(x_j) \ = \ \frac{\mathbf{U}(x_{j+1}) - \mathbf{U}(x_j)}{h_{j+1}} \quad \text{and} \quad D^-\mathbf{U}(x_j) \ = \ \frac{\mathbf{U}(x_j) - \mathbf{U}(x_{j-1})}{h_j}.$$

with $\overline{h}_j \ = \ \dfrac{h_j + h_{j+1}}{2}, \quad h_j \ = \ x_j - x_{j-1}$.
The following discrete results are analogous to those for the continuous case.

Lemma 7. *Let $A(x)$ satisfy (2) and (3). Then, for any mesh function $\mathbf{\Psi}$, the inequalities* $\mathbf{\Psi}(0) \geq \mathbf{0}$, $\mathbf{\Psi}(1) \geq \mathbf{0}$ and $\mathbf{L}^{\mathrm{N}}\mathbf{\Psi}(\mathrm{x_j}) \geq \mathbf{0}$ *for* $1 \leq j \leq N-1$, *imply that* $\mathbf{\Psi}(x_j) \geq \mathbf{0}$ *for* $0 \leq j \leq N$.

Proof. Let i^*, j^* be such that $\Psi_{i^*}(x_{j^*}) = \min_{i,j} \Psi_i(x_j)$ and assume that the lemma is false. Then $\Psi_{i^*}(x_{j^*}) < 0$. From the hypotheses we have $j^* \neq 0,\ N$ and $\Psi_{i^*}(x_{j^*}) - \Psi_{i^*}(x_{j^*-1}) \leq 0$, $\Psi_{i^*}(x_{j^*+1}) - \Psi_{i^*}(x_{j^*}) \geq 0$, so $\delta^2 \Psi_{i*}(x_{j*}) > 0$. It follows that

$$\left(\mathbf{L}^N \mathbf{\Psi}(x_{j*})\right)_{i*} = -\varepsilon_{i*}\delta^2 \Psi_{i*}(x_{j*}) + \sum_{k=1}^{n} a_{i*,\,k}(x_{j*})\Psi_k(x_{j*}) < 0,$$

which is a contradiction, as required.

An immediate consequence of this is the following discrete stability result.

Lemma 8. *Let $A(x)$ satisfy (2) and (3). Then, for any mesh function $\mathbf{\Psi}$,*

$$\| \mathbf{\Psi}(x_j) \| \leq \max \left\{ ||\mathbf{\Psi}(0)||, ||\mathbf{\Psi}(1)||, \frac{1}{\alpha}||\mathbf{L}^N \mathbf{\Psi}|| \right\},\ 0 \leq j \leq N.$$

Proof. Define the two functions

$$\mathbf{\Theta}^{\pm}(x_j) = \max\{||\mathbf{\Psi}(0)||, ||\mathbf{\Psi}(1)||, \frac{1}{\alpha}||\mathbf{L}^{\mathbf{N}}\mathbf{\Psi}||\}\mathbf{e} \pm \mathbf{\Psi}(x_j)$$

where $\mathbf{e} = (1, \ \ldots \ , 1)$ is the unit vector. Using the properties of A it is not hard to verify that $\mathbf{\Theta}^{\pm}(0) \geq \mathbf{0}$, $\mathbf{\Theta}^{\pm}(1) \geq \mathbf{0}$ and $\mathbf{L}^{\mathbf{N}}\mathbf{\Theta}^{\pm}(x_j) \geq \mathbf{0}$. It follows from Lemma 7 that $\mathbf{\Theta}^{\pm}(x_j) \geq \mathbf{0}$ for all $0 \leq j \leq N$.

4 The local truncation error

From Lemma 8, it is seen that in order to bound the error $||\mathbf{U} - \mathbf{u}||$ it suffices to bound $\mathbf{L}^N(\mathbf{U} - \mathbf{u})$. But this expression satisfies

$$\mathbf{L}^N(\mathbf{U} - \mathbf{u}) = \mathbf{L}^N(\mathbf{U}) - \mathbf{L}^N(\mathbf{u}) = \mathbf{f} - \mathbf{L}^N(\mathbf{u}) = \mathbf{L}(\mathbf{u}) - \mathbf{L}^N(\mathbf{u})$$

$$= (\mathbf{L} - \mathbf{L}^N)\mathbf{u} = -E(\delta^2 - D^2)\mathbf{u}$$

which is the local truncation of the second derivative. Then

$$E(\delta^2 - D^2)\mathbf{u} = E(\delta^2 - D^2)\mathbf{v} + E(\delta^2 - D^2)\mathbf{w}$$

and so, by the triangle inequality,

$$\| \mathbf{L}^N(\mathbf{U} - \mathbf{u}) \| \leq \| E(\delta^2 - D^2)\mathbf{v} \| + \| E(\delta^2 - D^2)\mathbf{w} \| . \tag{24}$$

Thus, the smooth and singular components of the local truncation error can be treated separately. In view of this it is noted that, for any smooth function ψ, the following two distinct estimates of the local truncation error of its second derivative hold

$$|(\delta^2 - D^2)\psi(x_j)| \le 2 \max_{s \,\in\, I_j} |\psi''(s)| \tag{25}$$

and

$$|(\delta^2 - D^2)\psi(x_j)| \;\le\; \frac{\delta_j}{3} \max_{s \,\in\, I_j} |\psi'''(s)| \tag{26}$$

where $I_j = [x_{j-1}, x_{j+1}]$.

5 Error estimate

The smooth component of the local truncation error is estimated in the following lemma.

Lemma 9. *Let $A(x)$ satisfy (2) and (3). Then, for each $i = 1, \ldots, n$ and $j = 1, \ldots, N-1$, we have*

$$|\varepsilon_i(\delta^2 - D^2)v_i(x_j)| \le C\sqrt{\varepsilon_i}N^{-1}.$$

Proof. Using (26), Lemma 4 and (20) it follows that

$$|\varepsilon_i(\delta^2 - D^2)v_i(x_j)| \;\le\; C\delta_j \max_{s \,\in\, I_j} |\varepsilon_i v_i'''(s)| \;\le\; C\sqrt{\varepsilon_i}\delta_j \;\le\; C\sqrt{\varepsilon_i}N^{-1}$$

as required. ∎

For the singular component a similar estimate is needed, but in the proof the different types of mesh must be distinguished. The following preliminary lemmas are required.

Lemma 10. *Let $A(x)$ satisfy (2) and (3). Then, for each $i = 1, \ldots, n$ and $j = 1, \ldots, N$, on each mesh $M_{\mathbf{b}}$, the following estimate holds*

$$|\varepsilon_i(\delta^2 - D^2)w_i^l(x_j)| \;\le\; \frac{C\delta_j}{\sqrt{\varepsilon_1}}.$$

Proof. From (26) and Lemma 5, it follows that

$$\begin{aligned}|\varepsilon_i(\delta^2 - D^2)w_i^l(x_j)| &\le C\delta_j \max_{s \,\in\, I_j} |\varepsilon_i w_i^{l,'''}(s)| \\ &\le C\delta_j \max_{s \,\in\, I_j} \sum_{q\,=\,1}^{n} \frac{B_q^l(s)}{\sqrt{\varepsilon_q}} \\ &\le \frac{C\delta_j}{\sqrt{\varepsilon_1}}\end{aligned}$$

as required. ∎

In what follows second degree polynomials of the form

$$p_{i;\theta}(x) = \sum_{k=0}^{2} \frac{(x-x_\theta)^k}{k!} w_i^{l,(k)}(x_\theta)$$

are used, where θ denotes a pair of integers separated by a comma.

Lemma 11. *Let $A(x)$ satisfy (2) and (3). Then, for each $i = 1, \ldots, n$, $j = 1, \ldots, N$ and $k = 1, \ldots, n-1$, on each mesh $M_{\mathbf{b}}$ with $b_k = 1$, there exists a decomposition*

$$w_i^l = \sum_{m=1}^{k+1} w_{i,m},$$

for which the following estimates hold for each m, $1 \le m \le k$,

$$|\varepsilon_i w''_{i,m}(x)| \le C B_m^l(x), \quad |\varepsilon_i w'''_{i,m}(x)| \le C \frac{B_m^l(x)}{\sqrt{\varepsilon_m}}$$

and

$$|\varepsilon_i w'''_{i,k+1}(x)| \le C \sum_{q=k+1}^{n} \frac{B_q^l(x)}{\sqrt{\varepsilon_q}}.$$

Furthermore

$$|\varepsilon_i(\delta^2 - D^2) w_i^l(x_j)| \le C(B_k^l(x_{j-1}) + \frac{\delta_j}{\sqrt{\varepsilon_{k+1}}}).$$

Analogous results hold for the w_i^r and their derivatives.

Proof. Since $b_k = 1$ it follows that $\sqrt{\varepsilon_k} \le \sqrt{\varepsilon_{k+1}}/2$, so $x_{k,k+1} \in (0, \frac{1}{2})$ and the decomposition

$$w_i^l = \sum_{m=1}^{k+1} w_{i,m},$$

exists, where the components of the decomposition are defined by

$$w_{i,k+1} = \begin{cases} p_{i;k,k+1} & \text{on } [0, x_{k,k+1}) \\ w_i^l & \text{otherwise} \end{cases}$$

and for each m, $k \ge m \ge 2$,

$$w_{i,m} = \begin{cases} p_{i;m-1,m} & \text{on } [0, \mathrm{x_{m-1,m}}) \\ w_i^l - \sum_{q=m+1}^{k+1} w_{i,q} & \text{otherwise} \end{cases}$$

and

$$w_{i,1} = w_i^l - \sum_{q=2}^{k+1} w_{i,q} \text{ on } [0,1].$$

From the above definitions it follows that, for each m, $1 \leq m \leq k$, $w_{i,m} = 0$ on $[\mathrm{x_{m,m+1}}, 1]$.
To establish the bounds on the third derivatives it is seen that:
for $x \in [x_{k,k+1}, 1]$, Lemma 5 and $x \geq x_{k,k+1}$ imply that

$$|\varepsilon_i w'''_{i,k+1}(x)| = |\varepsilon_i w_i^{l,'''}(x)| \leq C \sum_{q=1}^{n} \frac{B_q^l(x)}{\sqrt{\varepsilon_q}} \leq C \sum_{q=k+1}^{n} \frac{B_q^l(x)}{\sqrt{\varepsilon_q}};$$

for $x \in [0, x_{k,k+1}]$, Lemma 5 and $x \leq x_{k,k+1}$ imply that

$$|\varepsilon_i w'''_{i,k+1}(x)| = |\varepsilon_i w_i^{l,'''}(x_{k,k+1})| \leq \sum_{q=1}^{n} \frac{B_q^l(x_{k,k+1})}{\sqrt{\varepsilon_q}} \leq \sum_{q=k+1}^{n} \frac{B_q^l(x_{k,k+1})}{\sqrt{\varepsilon_q}} \leq \sum_{q=k+1}^{n} \frac{B_q^l(x)}{\sqrt{\varepsilon_q}};$$

and for each $m = k, \ \ldots \ ,2$, it follows that

for $x \in [x_{m,m+1}, 1]$, $w'''_{i,m} = 0$;
for $x \in [x_{m-1,m}, x_{m,m+1}]$, Lemma 5 implies that

$$|\varepsilon_i w'''_{i,m}(x)| \leq |\varepsilon_i w_i^{l,'''}(x)| + \sum_{q=m+1}^{k+1} |\varepsilon_i w'''_{i,q}(x)| \leq C \sum_{q=1}^{n} \frac{B_q^l(x)}{\sqrt{\varepsilon_q}} \leq C \frac{B_m^l(x)}{\sqrt{\varepsilon_m}};$$

for $x \in [0, x_{m-1,m}]$, Lemma 5 and $x \leq x_{m-1,m}$ imply that

$$|\varepsilon_i w'''_{i,m}(x)| = |\varepsilon_i w_i^{l,'''}(x_{m-1,m})| \leq C \sum_{q=1}^{n} \frac{B_q^l(x_{m-1,m})}{\sqrt{\varepsilon_q}} \leq C \frac{B_m^l(x_{m-1,m})}{\sqrt{\varepsilon_m}} \leq C \frac{B_m^l(x)}{\sqrt{\varepsilon_m}};$$

for $x \in [x_{1,2}, 1]$, $\ w'''_{i,1} = 0$;
for $x \in [0, x_{1,2}]$, Lemma 5 implies that

$$|\varepsilon_i w'''_{i,1}(x)| \leq |\varepsilon_i w_i^{l,'''}(x)| + \sum_{q=2}^{k+1} |\varepsilon_i w'''_{i,q}(x)| \leq C \sum_{q=1}^{n} \frac{B_q^l(x)}{\sqrt{\varepsilon_q}} \leq C \frac{B_1^l(x)}{\sqrt{\varepsilon_1}}.$$

For the bounds on the second derivatives note that, for each m, $1 \leq m \leq k$
:
for $x \in [x_{m,m+1}, 1]$, $\ w''_{i,m} = 0$;
for $x \in [0, x_{m,m+1}]$, $\ \int_x^{x_{m,m+1}} \varepsilon_i w'''_{i,m}(s) ds = \varepsilon_i w''_{i,m}(x_{m,m+1}) - \varepsilon_i w''_{i,m}(x) = -\varepsilon_i w''_{i,m}(x)$
and so

$$|\varepsilon_i w''_{i,m}(x)| \leq \int_x^{x_{m,m+1}} |\varepsilon_i w'''_{i,m}(s)| ds \leq \frac{C}{\sqrt{\varepsilon_m}} \int_x^{x_{m,m+1}} B_m^l(s) ds \leq C B_m^l(x).$$

Finally, since

$$|\varepsilon_i(\delta^2 - D^2) w_i^l(x_j)| \leq \sum_{m=1}^{k} |\varepsilon_i(\delta^2 - D^2) w_{i,m}(x_j)| + |\varepsilon_i(\delta^2 - D^2) w_{i,k+1}(x_j)|,$$

using (26) on the last term and (25) on all other terms on the right hand side, it follows that

$$|\varepsilon_i(\delta^2 - D^2)w_i^l(x_j)| \le C(\sum_{m=1}^{k} \max_{s\in I_j} |\varepsilon_i w''_{i,m}(s)| + \delta_j \max_{s\in I_j} |\varepsilon_i w'''_{i,k+1}(s)|).$$

The desired result follows by applying the bounds on the derivatives in the first part of this lemma.
The proof for the w_i^r and their derivatives is similar.

Lemma 12. *Let $A(x)$ satisfy (2) and (3). Then, for each $i = 1, \ldots, n$ and $j = 1, \ldots, N$, on each mesh $M_{\mathbf{b}}$ the following estimate holds*

$$|\varepsilon_i(\delta^2 - D^2)w_i^l(x_j)| \le CB_n^l(x_{j-1}).$$

An analogous result holds for w^r.

Proof. From (25) and Lemma 5, for each $i = 1, \ldots, n$ and $j = 1, \ldots, N$, it follows that

$$|\varepsilon_i(\delta^2 - D^2)w_i^l(x_j)| \le C \max_{s\in I_j} |\varepsilon_i w_i^{l,\prime\prime}(s)|$$

$$\le C\ \varepsilon_i \sum_{p=i}^{n} \frac{B_p^l(x_{j-1})}{\varepsilon_p} \le CB_n^l(x_{j-1}). \blacksquare$$

Using the above preliminary lemmas on appropriate subintervals, the desired estimate of the singular components of the local truncation error are proved in the following lemma.

Lemma 13. *Let $A(x)$ satisfy (2) and (3). Then, for each $i = 1, \ldots, n$ and $j = 1, \ldots, N$, the following estimate holds*

$$|\varepsilon_i(\delta^2 - D^2)w_i(x_j)| \le CN^{-1}\ln N.$$

Proof. Since $\mathbf{w} = \mathbf{w}^l + \mathbf{w}^r$, it suffices to prove the result for $\mathbf{w}^l$ and $\mathbf{w}^r$ separately. Here it is proved for $\mathbf{w}^l$. A similar proof holds for $\mathbf{w}^r$.
Stepping out from the origin each subinterval is treated separately.
First, consider $x \in (0, \sigma_1)$. Then, on each mesh $M_{\mathbf{b}}$, $\delta_j \le CN^{-1}\sigma_1$ and the result follows from (21) and Lemma 10.
Secondly, consider $x \in (\sigma_1, \sigma_2)$, then $\sigma_1 \le x_{j-1}$ and $\delta_j \le CN^{-1}\sigma_2$. The 2^{n+1} possible meshes are divided into 2 subclasses. On the meshes $M_{\mathbf{b}}$ with $b_1 = 0$ the result follows from (21), (18) and Lemma 10. On the meshes $M_{\mathbf{b}}$ with $b_1 = 1$ the result follows from (21), (19) and Lemma 11. When $x = \sigma_1$, similar arguments apply for the 2 subclasses, except that (22)is also needed for the second subclass.
Thirdly, in the general case $x \in (\sigma_m, \sigma_{m+1})$ for $2 \le m \le n-1$, it follows that $\sigma_m \le x_{j-1}$ and $\delta_j \le CN^{-1}\sigma_{m+1}$. Then $M_{\mathbf{b}}$ is divided into 3 subclasses:

$M_{\mathbf{b}}^0 = \{M_{\mathbf{b}} : b_1 = \cdots = b_m = 0\}$, $M_{\mathbf{b}}^r = \{M_{\mathbf{b}} : b_r = 1,\ b_{r+1} = \cdots = b_m = 0$ for some $1 \le r \le m-1\}$ and $M_{\mathbf{b}}^m = \{M_{\mathbf{b}} : b_m = 1\}$. On $M_{\mathbf{b}}^0$ the result follows from (21), (18) and Lemma 10; on $M_{\mathbf{b}}^r$ from (21), (18), (19) and Lemma 11; on $M_{\mathbf{b}}^m$ from (21), (19) and Lemma 11. When $x = \sigma_m$, similar arguments apply for the 3 subclasses, except that (22)is also needed for the third subclass.
Finally, for $x \in (\sigma_n, 1)$, $\sigma_n \le x_{j-1}$ and $\delta_j \le CN^{-1}$. Then $M_{\mathbf{b}}$ is divided into 3 subclasses: $M_{\mathbf{b}}^0 = \{M_{\mathbf{b}} : b_1 = \cdots = b_n = 0\}$, $M_{\mathbf{b}}^r = \{M_{\mathbf{b}} : b_r = 1,\ b_{r+1} = \cdots = b_n = 0$ for some $1 \le r \le n-1\}$ and $M_{\mathbf{b}}^n = \{M_{\mathbf{b}} : b_n = 1\}$. On $M_{\mathbf{b}}^0$ the result follows from (21), (18) and Lemma 10; on $M_{\mathbf{b}}^r$ from (21), (18), (19) and Lemma 11; on $M_{\mathbf{b}}^n$ from (19) and Lemma 12. When $x = \sigma_n$, similar arguments apply for the 3 subclasses, except that (22)is also needed for the third subclass. ∎

Let $\mathbf{u}$ denote the exact solution from (1) and $\mathbf{U}$ the discrete solution from (23). Then, the main result of this paper is the following parameter uniform error estimate

Theorem 1. *Let $A(x)$ satisfy (2) and (3). Then there exists a constant C such that*

$$\| \mathbf{U} - \mathbf{u} \| \le CN^{-1} \ln N,$$

for all $N > 1$.

Proof. This follows immediately by applying Lemmas 9 and 13 to (24) and using Lemma 8. ∎

6 An essentially second order method

In this section it is shown that a simple modification to the Shishkin mesh constructed above leads to an essentially second order parameter-uniform numerical method for (1). The finite difference operator is the same as for the first order method; the Shishkin piecewise uniform mesh is modified by choosing a different set of transition parameters. Instead of (16) and (17) the following parameters are used

$$\tau_n = \min\left\{\frac{1}{4}, 2\sqrt{\frac{\varepsilon_n}{\alpha}} \ln N\right\} \tag{27}$$

and for $i = 1, \ \ldots \ , n-1$

$$\tau_i = \min\left\{\frac{\tau_{i+1}}{2}, 2\sqrt{\frac{\varepsilon_i}{\alpha}} \ln N\right\}. \tag{28}$$

The proof that the resulting numerical method is essentially second order parameter-uniform is similar to the above and is based on an extension of the techniques employed in [3]. It is assumed henceforth that the problem data satisfy additional smoothness conditions, as required.

It is first noted that, with the definitions (27), (28), (19) is replaced by

$$B_i^l(\tau_i) = B_i^r(1-\tau_i) = N^{-2} \text{ when } b_i = 1. \tag{29}$$

Also, for any smooth function ψ, in addition to (25) and (26), the following estimate of the local truncation error holds at any mesh point x_j of a locally uniform mesh

$$|(\delta^2 - D^2)\psi(x_j)| \;\le\; \frac{\delta_j^2}{12} \max_{s\,\in\, I_j} |\psi''''(s)| \;\text{ if }\; x_j - x_{j-1} = x_{j+1} - x_j. \tag{30}$$

From their construction, it is clear that the above Shishkin meshes are locally uniform everywhere, except at the points $x_j = \tau_k$ where $k \in I_{\mathbf{b}}$ and $I_{\mathbf{b}} = \{k : b_k = 1\}$.
To estimate the smooth component of the error, note that the estimate in Lemma 9 can be modified to

$$|L^N(V-v)_i(x_j)| \le \begin{cases} C\sqrt{\varepsilon_i}N^{-1} & \text{if } \; x_j \in \{\tau_k, 1-\tau_k\}, k \in I_{\mathbf{b}} \\ CN^{-2} & \text{otherwise.} \end{cases} \tag{31}$$

Now introduce the mesh function $\mathbf{\Phi}$ where, for each $1 \le i \le n$,

$$\Phi_i(x_j) = CN^{-2}(\theta(x_j)+1),$$

where $\theta = \sum_{k\in I_{\mathbf{b}}} \theta_k$ and, for $k \in I_{\mathbf{b}}$, θ_k is the piecewise constant polynomial

$$\theta_k(x) = \begin{cases} 0, & x \in [0, \tau_k) \\ 1, & x \in [\tau_k, 1-\tau_k] \\ 0, & x \in (1-\tau_k, 1] \end{cases}$$

.
Then

$$0 \le \Phi_i(x_j) \le CN^{-2}, \;\; 1 \le i \le n$$

and

$$(L^N\Phi(x_j))_i = CN^{-2}[-\varepsilon_i \delta^2 \theta(x_j) + \sum_{j=1}^{n} a_{i,j}(x_j)(\theta(x_j)+1)].$$

It follows that

$$(L^N\Phi(x_j))_i \ge \begin{cases} C'(\varepsilon_i + N^{-2}) & \text{if } \; x_j \in \{\tau_k, 1-\tau_k\}, k \in I_{\mathbf{b}} \\ C'N^{-2} & \text{otherwise.} \end{cases} \tag{32}$$

Considering the cases $\varepsilon_i \ge N^{-1}$ and $\varepsilon_i < N^{-1}$ separately, choosing C' sufficiently small, comparing (32) with (31) and applying the discrete maximum principle to the barrier functions

$$\mathbf{\Psi}^{\pm} = \mathbf{\Phi} \pm (\mathbf{V} - \mathbf{v})$$

gives the following estimate

$$||(\mathbf{V}-\mathbf{v})|| \le CN^{-2}. \tag{33}$$

To estimate the singular component of the error, note that the estimates in Lemmas 10 and 11 are modified, respectively, to

$$|\varepsilon_i(\delta^2 - D^2)w_i^l(x_j)| \ \le \ \frac{C\delta_j^2}{\varepsilon_1} \tag{34}$$

$$|\varepsilon_i(\delta^2 - D^2)w_i^l(x_j)| \le C(B_k^l(x_{j-1}) + \frac{\delta_j^2}{\varepsilon_{k+1}}) \tag{35}$$

Combining these with Lemma 12, and repeating the same for the w_i^r leads to the following estimate of the singular component of the local truncation error

$$|\varepsilon_i(\delta^2 - D^2)w_i(x_j)| \le C(N^{-1}\ln N)^2. \tag{36}$$

Application of Lemma 8 then gives

$$||(\mathbf{W}-\mathbf{w})|| \le C(N^{-1}\ln N)^2. \tag{37}$$

Combining (33) and (37) leads at once to the required essentially second order parameter-uniform error estimate

$$||(\mathbf{U}-\mathbf{u})|| \le C(N^{-1}\ln N)^2. \tag{38}$$

References

1. J. J. H. Miller, E. O'Riordan, G.I. Shishkin, *Fitted Numerical Methods for Singular Perturbation Problems*, World Scientific Publishing Co., Singapore, New Jersey, London, Hong Kong (1996).
2. H.-G. Roos and M. Stynes and L. Tobiska, Numerical methods for singularly perturbed differential equations, Springer Verlag, 1996.
3. J.J.H. Miller, E. O'Riordan, G.I. Shishkin and L.P. Shishkina, Fitted mesh methods for problems with parabolic boundary layers, Math. Proc. Roy. Irish Acad. Vol. 98A, No. 2, 173-190 (1998).
4. P.A. Farrell, A. Hegarty, J. J. H. Miller, E. O'Riordan, G. I. Shishkin, *Robust Computational Techniques for Boundary Layers*, Applied Mathematics & Mathematical Computation (Eds. R. J. Knops & K. W. Morton), Chapman & Hall/CRC Press (2000).
5. S. Matthews, J.J.H. Miller, E. O'Riordan and G.I. Shishkin, A parameter robust numerical method for a system of singularly perturbed ordinary differential equations, Nova Science Publishers, New York, 2000.
6. N. Madden, M. Stynes, A uniformly convergent numerical method for a coupled system of two singularly perturbed reaction-diffusion problems, IMA J. Num. Anal., 23, 627-644 (2003).
7. S.Hemavathi, S. Valarmathi, *A parameter uniform numerical method for a system of singularly perturbed ordinary differential equations*, Proceedings of the International Conference on Boundary and Interior Layers, BAIL 2006, Goettingen (2006).
8. T. Linss, N. Madden, Layer-adapted meshes for a linear system of coupled singularly perturbed reaction-diffusion problems, IMA J. Num. Anal., 29, 109-125 (2009).